

Probability Theory II
Basics of Stochastic Processes; Markov Processes and
Martingales

Seiji HIRABA

October 8, 2020

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In this text, we describe Markov properties and martingale properties on discrete/continuous-time stochastic processes

We give examples of Markov processes, random walks, Galton-Watson processes and Poisson processes, and we investigate their properties.